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RING INDEXED LIE ALGEBRAS

A tribute to L Onsager's Algebra underlying his celebrated 2d Ising model solution (1944).

Onsager's integer-indexed infinite-dimensional Lie Algebra of spin chain operators,

$$[A_m, A_n] = 4G_{m-n}, \qquad [G_m, A_n] = 2(A_{m+n} - A_{n-m}), \qquad [G_m, G_n] = 0.$$

$\rightsquigarrow \quad G_{-m} = -G_m.$

A potential central element, G_0, is not generated on the r.h.s. of the algebra.

Onsager also recognized that his algebra is effectively a subalgebra of the $SL(2)$ loop algebra ($SU(2)$ centerless Kac-Moody): The loop Lie algebra consists of three integer-indexed **towers** of elements, with

$$[K^+_m, K^-_n] = K^0_{m+n} \,, \quad [K^0_m, K^\pm_n] = \pm K^\pm_{m+n} \,, \quad [K^\pm_m, K^\pm_n] = [K^0_m, K^0_n] = 0.$$

Given the linear involutive automorphism of this algebra,

$$K^\pm_m \mapsto K^\mp_{-m} \,, \qquad\qquad K^0_m \mapsto -K^0_{-m} \,,$$

the Onsager algebra is identifiable with the fixed-point subalgebra (Uglov & Ivanov): the subalgebra invariant under this automorphism, consisting of two towers,

$$A_m = 2\sqrt{2}\,(K^+_m + K^-_{-m}), \qquad\qquad G_m = 2\,(K^0_m - K^0_{-m}).$$

Inclusion of the Kac-Moody center leads to no modifications.

Generalizes to $SL(N)$ (Uglov & Ivanov, 1996) nicely.

But, can we generalize beyond subalgebras of Kac-Moody?

Yes, new algebras, provided we consider interesting rings of indices:

$$\boxed{[J^a_m, J^b_n] = J^{a+b}_{m+\omega^a n} - J^{a+b}_{n+\omega^b m}} \, ,$$

where the indices $a, b, ...,\ m, n, ...$ and the parameter ω may be arbitrary, in general.

However, the choice of ω as an N-th root of unity, $\omega^N = 1$, hence $1 + \omega + \omega^2 + ... + \omega^{N-1} = 0$, and $a, b, ...$ integers, $m, n, ...$ proportional to integers, yields by far the most interesting family.

$\rightsquigarrow$ the upper indices are only distinct mod$N$; and the lower indices take values in the cyclotomic integer ring $\mathbb{Z}[\omega]$, namely, $r + s\omega + k\omega^2 + ... + j\omega^{N-2}$.

NB Grading of the upper indices, but lack of conventional grading for the lower indices.

NB Satisfies the Jacobi identity.

- Possesses the **central element**

$$J^0_0 = J^{-a}_{-\omega^{-a}m} J^a_m .$$

For the cyclotomic family, "Casimir invariants" may be written as

$$J^0_0 = (J^a_m)^N ,$$

provided $m = 0$ if $a = 0$.

In fact, this Lie algebra might be constructed from the group algebra of associative operators

$$J^a_m J^b_n = J^{a+b}_{m+\omega^a n} \qquad\qquad (= J^b_{\omega^a n} J^a_{\omega^{-b} m}),$$

which satisfy $(J^a_m J^b_n) J^c_k = J^a_m (J^b_n J^c_k)$.

$\rightsquigarrow$ Customary in such cases to also consider the anticommutator of these operators, to produce a **partner graded Lie algebra**,

$$\{J^a_m, J^b_n\} = J^{a+b}_{m+\omega^a n} + J^{a+b}_{n+\omega^b m}\ .$$

• A simple **operator realization** of this algebra:

$$J^a_m = e^{m \exp(x)}\ \omega^{a\partial_x},$$

(Recall translation action of $\omega^{\partial_x} f(x) = f(x + \ln\omega)\ \omega^{\partial_x}$.)

$\rightsquigarrow$ Easy to see that the scale of the a, b is fixed, but that of the m, n is labile, as they can be rescaled with no change to the structure of the algebra.

• A variant rewriting of this realization results from the simplifying Campbell-Baker-Hausdorff expansion for the particular operators involved,

$$J^a_m = \omega^{a\left(\partial_x + \frac{m}{\omega^a - 1}\exp(x)\right)}.$$

$\rightsquigarrow$ Equivalently, given oscillator operators, $[\alpha, \alpha^\dagger] = 1$, the above realizations may be written in a form evocative of vertex operators,

$$J^a_m = e^{m\ \alpha^\dagger}\ \omega^{a\ \alpha^\dagger\alpha} = \omega^{a\left(\alpha^\dagger\alpha + \frac{m}{\omega^a - 1}\alpha^\dagger\right)}.$$

In the cyclotomic case, $\omega^N = 1$, a, b are equivalent mod$N$:
$a, b, .. = 0, 1, 2, ..., N-1$.

The $N = 2$ case, $\omega = -1$, $a = 0, 1$, is old news, as the corresponding lower index ring is that of the conventional integers; and the resulting algebra degenerates to essentially the Onsager algebra, a subalgebra of the $SL(2)$ loop algebra,

$$A_m = 2J^1_m \,, \qquad G_m = J^0_m - J^0_{-m} \,.$$

$\leadsto$ A graded extension of the Onsager algebra of this type is trivial, since

$$H_{m-n} \equiv \{A_m, A_n\} = 4(J^0_{m-n+} J^0_{n-m}),$$

check to be central—commute with all elements, A_n, G_n.
$\leadsto$ $J^0_m \sim -J^0_{-m}+$'constant'; hence, conversely, requiring a trivial graded extension of the Onsager algebra essentially amounts to the product algebra. (NB $A_m A_m$ is not an invariant of the Onsager algebra per se, but only upon this further condition, $A_m A_m = 4J^0_0$.)

• Above realization reduces here to

$$A_m = 2e^{m\exp(x)}(-)^{\partial_x}, \qquad G_m = e^{m\exp(x)} - e^{-m\exp(x)}.$$

In this realization, the potential candidate for a graded extension,

$$H_m = 4(e^{m\exp(x)} + e^{-m\exp(x)}),$$

manifestly commutes with all elements, A_n, G_n .

• An alternate realization in terms of Pauli matrices is

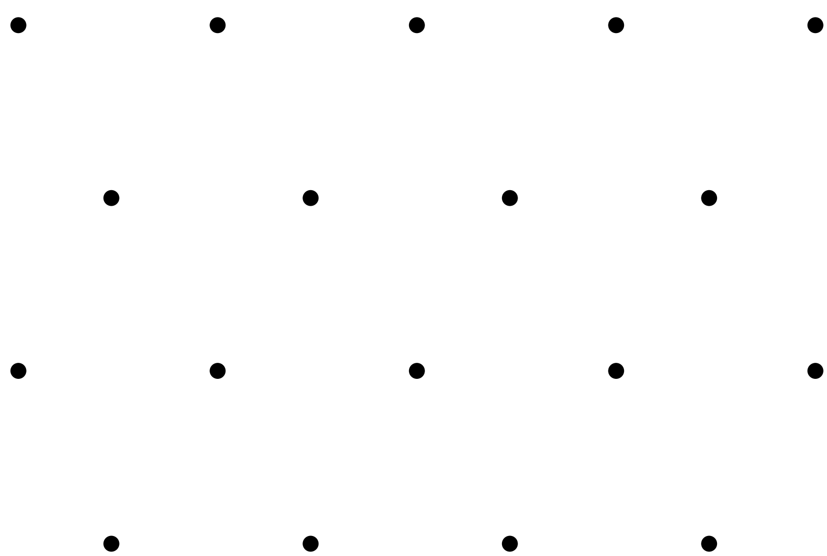
$$A_m = 2e^{m\sigma_3}\sigma_1 \,, \qquad G_m = (e^m - e^{-m})\sigma_3 \,,$$

similarly illustrating the triviality of $H_m \propto \mathbb{1}$.

THE EISENSTEIN INTEGER LATTICE

$\omega = e^{2\pi i/3} = -1 - \omega^2$, so the lower indices are of the form $m \equiv k + j\omega$ (with integer k, j), closing under addition, subtraction, and multiplication.

$\rightsquigarrow$ Comprise the Euclidean ring $\mathbb{Z}[\omega]$ of Eisenstein-Jacobi integers. These define a triangular 2-d lattice with hexagonal rotational symmetry: there are three lines at 60° to each other going through each such integer and connecting it to its six nearest neighbors, forming honeycomb hexagons:



Lattice of utility in cohesive energy calculations for monolayer graphite, 3-state-Potts models associated with WZW CFT models, and, perhaps more provocatively, complexifies to define the complex Leech lattice, of significance in string theory, and $\mathbb{Z}_3$ orbifolds in CFT.

Each point on the lattice may be connected to the origin by shifts along the ω root and along the x-axis. A 60^0 rotation ωm, on $m \equiv k + j\omega$, for integer coordinates k, j, may be represented by

$$\mathbf{\Omega} \begin{pmatrix} k \\ j \end{pmatrix} \equiv \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k \\ j \end{pmatrix},$$

for $\mathbf{\Omega}^3 = 1\!\!1$, and $\mathbf{\Omega}^2 = -1\!\!1 - \mathbf{\Omega}$.
Thus, the lower indices of the algebra may be considered as a **doublet of integers** composing through this rule.

$\rightsquigarrow$ Illustrate explicitly to stress the differences from conventional loop algebras and $sl(3)$ generalizations of the Onsager algebra.

Faithful representation in terms of 3×3 matrices. Sylvester's "nonion" basis for $GL(3)$ groups (1882), is built out of his standard clock and shift unitary unimodular matrices,

$$Q \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad P \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

so that $Q^3 = P^3 = 1\!\!1$.

These obey the braiding identity $PQ = \omega\, QP$.

For integer indices adding mod 3, the complete set of nine unitary unimodular 3×3 matrices

$$M_{(m_1,m_2)} \equiv \omega^{m_1 m_2/2}\, Q^{m_1} P^{m_2},$$

where $M^\dagger_{(m_1,m_2)} = M_{(-m_1,-m_2)}$, and $\text{Tr} M_{(m_1,m_2)} = 0$, except for $m_1 = m_2 = 0$ mod3, suffice to span the group algebra of $GL(3)$.

Further, since

$$M_{\mathbf{m}} M_{\mathbf{n}} = \omega^{\mathbf{n}\times\mathbf{m}/2} M_{\mathbf{m}+\mathbf{n}} ~,$$

where $\mathbf{m}\times\mathbf{n} \equiv m_1 n_2 - m_2 n_1$, they also satisfy the Lie algebra of $su(3)$ (FFZ),

$$[M_{\mathbf{m}}, M_{\mathbf{n}}] = -2i ~\sin\left(\frac{\pi}{3}\mathbf{m}\times\mathbf{n}\right) ~M_{\mathbf{m}+\mathbf{n}} ~.$$

It is then simple to realize the algebra in the unimodular 3×3 matrix representation,

$$\boxed{J^a_m = e^{mQ} ~P^a,} ~,$$

ie, the three "towers",

$$J^1_m = \begin{pmatrix} 0 & e^m & 0 \\ 0 & 0 & e^{m\omega} \\ e^{m\omega^2} & 0 & 0 \end{pmatrix}, J^2_m = \begin{pmatrix} 0 & 0 & e^m \\ e^{m\omega} & 0 & 0 \\ 0 & e^{m\omega^2} & 0 \end{pmatrix}, J^0_m = \begin{pmatrix} e^m & 0 & 0 \\ 0 & e^{m\omega} & 0 \\ 0 & 0 & e^{m\omega^2} \end{pmatrix}.$$

Contrast this Lie algebra to not only $su(3)$ loop algebra, but also to its subalgebras, such as the the $sl(3)$ generalization of the Onsager algebra (Uglov & Ivanov) consisting of five towers. The relevant involutive automorphism of $su(3)$ loop algebra, in standard Chevalley notation, is

$$H_m^{1,2} \mapsto -H_{-m}^{1,2}\,, \quad E_m^{\pm 1} \mapsto E_{-m}^{\mp 1}\,, \quad E_m^{\pm 2} \mapsto E_{-m}^{\mp 2}\,, \quad E_m^{\pm 3} \mapsto -E_{-m}^{\mp 3}\,.$$

The subalgebra left invariant under this automorphism consists of the five towers,

$$H_m^{1,2} - H_{-m}^{1,2}\,, \qquad E_m^{1} + E_{-m}^{-1}\,, \qquad E_m^{2} + E_{-m}^{-2}\,, \qquad E_m^{3} - E_{-m}^{-3}\,,$$

or, explicitly,

$$h_m^1 = \frac{1}{\sqrt{6}} \begin{pmatrix} e^m - e^{-m} & 0 & 0 \\ 0 & e^{-m} - e^m & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$h_m^2 = \frac{1}{3\sqrt{2}} \begin{pmatrix} e^m - e^{-m} & 0 & 0 \\ 0 & e^m - e^{-m} & 0 \\ 0 & 0 & 2e^{-m} - 2e^m \end{pmatrix},$$

$$e_m^1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & e^m & 0 \\ e^{-m} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_m^2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & e^m \\ 0 & 0 & 0 \\ e^{-m} & 0 & 0 \end{pmatrix},$$

$$e_m^3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^m \\ 0 & -e^{-m} & 0 \end{pmatrix}.$$

• For higher $N \geq 5$, the cyclotomic integer rings $\mathbb{Z}[\omega]$ are less compelling, and are linked to quasicrystals. Specifically, the 2-dim complex plane $\mathbb{R}^2$ fills up densely with the quasilattice set of indices, which fail to close to a "sparse" periodic structure analogous to the Eisenstein lattice.

E.g., for $N = 5$, motions are symmetric on a 4-dimensional periodic lattice, $\Omega^5 = \mathbb{1}$, and $\mathbf{\Omega}^4 = -\mathbb{1} - \mathbf{\Omega} - \mathbf{\Omega}^2 - \mathbf{\Omega}^3$, with

$$\mathbf{\Omega} \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

so lower indices may be effectively regarded as a quartet of integers—and, likewise, an $N-1$-tuplet of integers for higher N. However, projected on the actual complex plane, nearby numbers are not necessarily represented by contiguous points on the 4-d lattice.

• A quasicrystal is a higher-dimensional deterministic discrete periodic structure whose projection to an embedded "external space" (in our case, the complex plane) yields nonperiodic structures of enhanced regularity.

Links between these algebras over cyclotomic fields and those on quasicrystals which exhibit a five-fold symmetry. For $\omega^5 = 1$ and the golden ratio, $\tau \equiv \frac{1}{2}(1 + \sqrt{5}) \sim 1.618$, which satisfies $\tau^2 = 1 + \tau$, note that $\tau = -\omega^2 - \omega^3$, since then $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ follows.

The field $\mathbb{Z}[\tau]=\mathbb{Z}+\mathbb{Z}\tau$ suitably restriced to the Fibonacci chain, $n\tau+\left\lfloor \frac{n}{\tau}+1\right\rfloor$, an aperiodic point set of 1-d cut-and-project quasicrystals, which is **not a ring**.

Nevertheless, FTZ have found an associative graded composition in a self-selecting subset of it. So this one can still serve as the domain of lower indices in a extension of these algebras to a structure which, improbably, still respects associativity!

• Fruitful source of insight.
Vertex operator realization of the Lie algebras introduced and its evocation of coherent states,
$\rightsquigarrow$ likely useful applications in CFT and brane physics.